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PAPER: Quantum statistical physics, condensed matter, integrable systems

The Weyl–Heisenberg ensemble: hyperuniformity and higher Landau levels

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Abstract. Weyl–Heisenberg ensembles are a class of determinantal point processes associated with the Schrödinger representation of the Heisenberg group. Hyperuniformity characterizes a state of matter for which (scaled) density fluctuations diminish towards zero at the largest length scales. We will prove that Weyl–Heisenberg ensembles are hyperuniform. Weyl–Heisenberg ensembles include as a special case a multi-layer extension of the Ginibre ensemble modeling the distribution of electrons in higher Landau levels, which has recently been object of study in the realm of the Ginibre-type ensembles associated with polyanalytic functions. In addition, the family of Weyl–Heisenberg ensembles includes new structurally anisotropic processes, where point-statistics depend on the different spatial directions, and thus provide a first means to study directional hyperuniformity.

Keywords: correlation functions, fluctuation phenomena, matrix models, quantum chaos

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1. Introduction

The characterization of density fluctuations in many-body systems is a problem of great interest in the physical, mathematical and biological sciences. A *hyperuniform* many-particle system is one in which density fluctuations are anomalously suppressed at long-wavelengths, compared to those occurring in the Poisson point process and typical correlated disordered point processes. The hyperuniformity concept provides a new way to classify crystals, certain quasiperiodic systems, and special disordered systems [1, 2]. A theory for understanding hyperuniformity in terms of the number variance of point processes has been developed in [1] and [3]. The theory characterizes hyperuniform point process in d -dimensions with the property that the variance in the number of points in an observation window of radius R grows at a rate slower than R^d , which is the growth rate for a Poisson point process.

It is known that some determinantal point processes are disordered and hyperuniform [4–8]. *Weyl–Heisenberg Ensembles* are a very general class of determinantal point processes on \mathbb{R}^d , with $d = 2m$ an even number. They are defined in terms of the Schrödinger representation of the Heisenberg group acting on a vector $g \in L^2(\mathbb{R}^m)$. For choices of g in the Hermite function basis of $L^2(\mathbb{R}^m)$, they reduce to extensions of the

two-dimensional one-component plasma and the Ginibre ensemble to higher Landau levels. In this paper, we will show that Weyl–Heisenberg ensembles are hyperuniform. Actually, a bit more is true: the variance of the number of points in an observation window of radius R grows at a rate proportional to R^{d-1} . This was already known to happen in the two-dimensional one-component plasma. Our results about the Weyl–Heisenberg ensembles show that the same happens with the distribution of electrons when higher Landau levels are formed under strong magnetic fields. Higher Landau levels lead to the macroscopic effect known as the quantum-Hall effect [9, 10].

There are several ways of randomly distributing points on the Euclidean space. Under mild assumptions, a point process is completely specified by the countably infinite set of generic k -particle probability density functions, denoted by $\rho_k(\mathbf{r}_1, \dots, \mathbf{r}_k)$. These are proportional to the probability density of finding k particles in volume elements around the given positions $(\mathbf{r}_1, \dots, \mathbf{r}_k)$, irrespective of the remaining particles. More precisely, if \mathcal{X} is a simple process defined on the Euclidean space—so that points do not have multiplicities, then ρ_k is characterized by the properties that (i) $\rho_k(\mathbf{r}_1, \dots, \mathbf{r}_k) = 0$, whenever the positions of two of the points \mathbf{r}_j are equal, and (ii) for every family of disjoint measurable sets D_1, \dots, D_k ,

$$\mathbb{E} \left[\prod_{j=1}^k \mathcal{X}(D_j) \right] = \int_{\prod_j D_j} \rho_k(\mathbf{r}_1, \dots, \mathbf{r}_k) d\mathbf{r}_1 \dots d\mathbf{r}_k,$$

where $\mathcal{X}(D)$ denotes the number of points to be found in D . For instance, Poisson processes distribute points randomly but in a completely uncorrelated way. Indeed, in this case, the k -particle probability density reduces to constants easy to compute. Since the single-particle density function can be obtained from the thermodynamical limit ρ , where V is the strength of the magnetic field and grows together with the number of points N :

$$\rho(\mathbf{r}_1) = \lim_{N, V \rightarrow \infty} \frac{N}{V} = \rho,$$

the k -particle probability densities for Poisson processes are given by

$$\rho_k(\mathbf{r}_1, \dots, \mathbf{r}_k) = \rho^k.$$

However, in several many-body systems and other physical models, one has to take into account particle-particle interactions, requiring more sophisticated probabilistic models. In studies of the statistical mechanics of point-like particles one is usually interested in a handful of quantities such as k -particle correlations. It is then of paramount importance to study point processes for which the properties of such correlations have convenient analytic descriptions as, for instance, in the so-called ghost random sequential addition processes [11]. But such exactly solvable models are not so common, leading to widespread use of Poisson processes instead of more sophisticated probability models, because simple analytic expressions for the k -particle correlations are available. Facing such a gap between the physical model and its mathematical description, one may be led to think that the possibility of using probabilistic models describing interacting particles, with k -particle correlations written in analytic form, is a hopeless mathematical chimera. However, using determinantal point processes, it is possible to construct such probabilistic models.

Determinantal point processes (DPP's) are defined in terms of a kernel where the negative correlation between points is built in. Because of the repulsion inherent of the model, they are convenient to describe physical systems with charged-like particles, where the confinement to a bounded region is controlled by a weight function involving the external field. Unlike other non-trivial statistical models, the k -particle correlations of DPP's admit an analytic expression as determinants whose entries are defined using the correlation kernel. Moreover, DPP's enjoy a remarkable property which allows one to derive the macroscopic laws of physical systems constituted by interacting particles which display chaotic random behavior at small scales: for very large systems of points confined to bounded regions, the distribution patterns begin to look less chaotic and start to organize themselves in an almost uniform way. In several cases, in the proper scaled thermodynamical limit, the distributions are either uniform or given by analytic expressions. This phenomenon is related to a property of physical and mathematical systems known as *universality* [12].

The present paper explores a link between the theory of DPP's and the Schrödinger representation of the Heisenberg group. This link is important because it allows one to deal with problems involving non-analytic functions of complex variables using real-variable methods. It has also been used in [13] to obtain analytic and probabilistic results for a large class of planar ensembles, building on previous work on time-frequency analysis and approximation theory [14, 15]. While the papers [13, 15] are concerned with universality-type limit distribution laws and with probabilistic aspects of finite-dimensional Weyl–Heisenberg ensembles, in this paper we will focus on *infinite Weyl–Heisenberg ensembles*.

The article is organized as follows. Section 2 introduces planar DPP's and our model process, the Ginibre ensemble. The notion of hyperuniformity is introduced in section 3. Section 4 presents higher Landau levels and the polyanalytic ensembles, that describe them mathematically. Section 5 presents Weyl–Heisenberg ensembles and the main results on hyperuniformity, which are then applied to the polyanalytic ensemble and higher Landau levels. We also discuss and analyze the total correlation function and the structure factor associated with these point processes, and provide explicit formulae whenever possible. Finally section 6 summarizes the results.

2. Planar DPP's and the Ginibre ensemble

Determinantal point processes are defined using an ambient space Λ and a Radon measure μ defined on Λ [16, 17]. In our case, $\Lambda = \mathbb{R}^d$, where $d = 2m$ is an even number, and we use the identification $\mathbb{R}^d = \mathbb{C}^m$, since several of the examples of interest are best described in terms of complex variables. For this reason, we sometimes denote points in $\mathbb{R}^d = \mathbb{C}^m$ by z , instead of the usual r . The important object is the Hilbert space $L^2(\mathbb{C}^m)$, with the Lebesgue measure in \mathbb{C}^m . If $\{\varphi_j(z)\}_{j \geq 0}$ is an orthogonal sequence of $L^2(\mathbb{C}^m)$, one can define a reproducing kernel $K_N(z, w)$ by writing

$$K_N(z, w) = \sum_{j=0}^{N-1} \varphi_j(z) \overline{\varphi_j(w)}. \quad (1)$$

The kernel $K_N(z, w)$ will be the correlation kernel of the point process \mathcal{X} , whose k -point intensities are given by $\rho_k(x_1, \dots, x_k) = \det(K_N(x_i, x_j))_{1 \leq i, j \leq k}$. For instance, if $m = 1$, selecting $\varphi_j(z) = (\pi^j/j!)^{\frac{1}{2}} e^{-\frac{\pi}{2}|z|^2} z^j$ for $j = 0, \dots, N - 1$ in (1), we obtain

$$K_N(z, w) = e^{-\frac{\pi}{2}(|z|^2 + |w|^2)} \sum_{j=0}^{N-1} \frac{(\pi z \bar{w})^j}{j!},$$

which is the correlation kernel of the *Ginibre ensemble* of dimension N . If we take $N \rightarrow \infty$, we obtain the correlation kernel of the infinite Ginibre ensemble:

$$K_\infty(z, w) = e^{\pi z \bar{w} - \frac{\pi}{2}(|z|^2 + |w|^2)}. \tag{2}$$

The infinite Ginibre ensemble is *translationally invariant*; this means that the intensity functions satisfy: $\rho_N(z_0 + z, \dots, z_{N-1} + z) = \rho_N(z_0, \dots, z_{N-1})$, for all $z \in \mathbb{C}$.

It is well known that the Ginibre ensemble is equivalent to a model for the probability distribution of electrons in one component plasmas [18]. It also provides a model for the statistical quantum dynamics of a charged particle evolving in a Euclidean space under the action of a constant homogeneous magnetic field in the first Landau level. The Ginibre ensemble can also be seen as a 2D electrostatic model with N unit charges interacting in a two dimensional space, which is taken as the complex plane of the variable z . Indeed, if the potential energy of the system is given as

$$U(z_0, \dots, z_{N-1}) = - \sum_{0 \leq i < j \leq N-1} \log |z_i - z_j| + \pi \sum_{k=0}^{N-1} |z_k|^2,$$

the corresponding probability distribution of the positions z_0, \dots, z_{N-1} when the charges are in thermodynamical equilibrium, is proportional to the measure

$$\exp[-U(z_0, \dots, z_{N-1})] = \exp \left[-\pi \sum_{k=0}^{N-1} |z_k|^2 \right] \prod_{0 \leq i < j \leq N-1} |z_i - z_j|^2. \tag{3}$$

It has been shown by Jean Ginibre [18, 19] that the distribution associated with the measure (3) is proportional to the one obtained from the N -point intensities associated with the Ginibre ensemble of dimension N . Thus, the Ginibre ensemble provides a model for the distribution of charged-like particles in the first Landau level. Until recently, there was no similar Ginibre type model for higher Landau levels, but this gap in the statistical physics literature is being filled thanks to recent work concerning the *polyanalytic Ginibre ensembles* [13, 20–23]—see also section 4.

3. Hyperuniformity of point processes

3.1. The number mean of a DPP

The number mean of a point process is the average number of points expected to be found inside an observation window $D \subset \mathbb{R}^n$. One can obtain the number mean by integrating the single particle probability density, which is proportional to the probability density of finding a particle at a certain point $r \in D$. In the case of a DPP, it can be

obtained from the 1-point intensity ρ , which is simply defined as the diagonal of the correlation kernel of the process:

$$\rho(\mathbf{r}) = K(\mathbf{r}, \mathbf{r}) = \sum_i |\varphi_i(\mathbf{r})|^2.$$

The expected number of points $\mathcal{X}(D)$ to be found in $D \subset \mathbb{C}$ is then given as

$$\mathbb{E}[\mathcal{X}(D)] = \int_D \rho(\mathbf{r}) d\mathbf{r}.$$

The 1-point intensity ρ is also called the single particle probability density.

3.2. Number variance and hyperuniformity

In [1], it has been discovered that a hyperuniform many-particle system modeled by a point process \mathcal{X} in a Euclidean space of dimension d (not necessarily determinantal) is one in which the number variance

$$\sigma^2(R) = \mathbb{E}[\mathcal{X}(D_R)^2] - \mathbb{E}[\mathcal{X}(D_R)]^2,$$

where D_R is a d -dimensional ball of radius R , satisfies

$$\sigma^2(R) = o(R^d).$$

3.3. The total correlation function

We will be mostly interested in a translationally invariant point process of intensity 1, i.e. the intensity functions satisfy $\rho_n(\mathbf{r}_1 + \mathbf{r}, \dots, \mathbf{r}_n + \mathbf{r}) = \rho_n(\mathbf{r}_1, \dots, \mathbf{r}_n)$, for all $\mathbf{r} \in \mathbb{R}^n$, and $\rho_1 \equiv 1$. For such processes, the two-point intensity depends essentially on one variable, and we may write:

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2) = 1 + h(\mathbf{r}_2 - \mathbf{r}_1), \quad (4)$$

where h is known as the *total correlation function*, and is related to the determinantal kernel by

$$|K(\mathbf{r}_1, \mathbf{r}_2)|^2 = -h(\mathbf{r}_2 - \mathbf{r}_1). \quad (5)$$

In statistical mechanics, it is also common to consider the *structure factor* defined by

$$S(\mathbf{k}) = 1 + \hat{h}(\mathbf{k}),$$

in the reciprocal space (Fourier) variable \mathbf{k} . (We normalize the Fourier transform as: $\hat{f}(\mathbf{k}) = \int f(\mathbf{r}) e^{i\mathbf{r} \cdot \mathbf{k}} d\mathbf{r}$.)

4. Polyanalytic Ginibre ensembles and higher Landau levels

4.1. The Landau levels

Polyanalytic ensembles of the pure type model the random distribution of charged-like electrons in the so-called *Landau levels*. Let us briefly describe this relation (see

[20, 24, 26] for more details). The Hamiltonian operator describing the dynamics of a particle of charge e and mass m_* on the Euclidean xy -plane, while interacting with a perpendicular constant homogeneous magnetic field, is given by the operator

$$H := \frac{1}{2m_*} \left(i\hbar\nabla - \frac{e}{c}\mathbf{A} \right)^2, \tag{6}$$

where \hbar denotes Planck’s constant, c is the light velocity and i the imaginary unit. Denote by $B > 0$ the strength of the magnetic field and select the symmetric gauge

$$\mathbf{A} = -\frac{\mathbf{r}}{2} \times \mathbf{B} = \left(-\frac{B}{2}y, \frac{B}{2}x \right),$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$. For simplicity, we set $m_* = e = c = \hbar = 1$ in (6), leading to the Landau Hamiltonian

$$H_B := \frac{1}{2} \left(\left(i\partial_x - \frac{B}{2}y \right)^2 + \left(i\partial_y + \frac{B}{2}x \right)^2 \right) \tag{7}$$

acting on the Hilbert space $L^2(\mathbb{R}^2, dx dy)$. The spectrum of the Hamiltonian H_B consists of an infinite number of eigenvalues with infinite multiplicity of the form

$$\epsilon_n^B = \left(n + \frac{1}{2} \right) B, \quad n = 0, 1, 2, \dots \tag{8}$$

Without loss of generality, we set $B = 2\pi$ to simplify the relation to the Weyl–Heisenberg group described in the next sections. Then we define the operator L_z by conjugating the Landau Hamiltonian (7) as follows:

$$L_z := e^{\frac{\pi}{2}|z|^2} \left(\frac{1}{2}H_{2\pi} - \frac{\pi}{2} \right) e^{-\frac{\pi}{2}|z|^2} = -\partial_z\partial_{\bar{z}} + \pi\bar{z}\partial_{\bar{z}}, \tag{9}$$

acting on the Hilbert space $L^2(\mathbb{C})$. The spectrum of L_z is given by $\sigma(L_z) = \{\nu\pi : \nu = 0, 1, 2, \dots\}$. The eigenvalue $r\pi$ is the Landau level of order r . The eigenspace associated with the eigenvalue $r\pi$ is called the *pure Landau level eigenspace of order r* . With each pure Landau level eigenspace of order r one can associate correlation kernels of the form

$$K_r(z, w) = L_r^0(\pi|z - w|^2) e^{\pi z\bar{w} - \frac{\pi}{2}(|z|^2 + |w|^2)}, \tag{10}$$

where L_r^0 is the Laguerre polynomial, defined, for a general parameter α , as

$$L_n^\alpha(x) = \sum_{k=0}^n (-1)^k \binom{n + \alpha}{n - k} \frac{x^k}{k!}.$$

As we will see in the next section, (10) is the reproducing kernel of a *pure Fock space of polyanalytic functions*. Thus, we name the resulting determinantal point process as a *polyanalytic ensemble of the pure type*. It is related to the polyanalytic Ginibre ensembles investigated in [20] and the terminology *pure type* is inherited from the Landau level interpretation: determinantal processes with kernels of the form considered in [20] have a physical interpretation as probabilistic 2D models for the distribution of

electrons in the first N Landau levels, while processes with correlation kernels of the form (10) model the distribution of electrons in a *pure* Landau level of order r . For reference, one can keep in mind that the basis functions of the Ginibre ensemble generate the proper subspace of $L^2(\mathbb{C})$ consisting of analytic functions (the so-called Bargmann–Fock space). Moreover, the case $r = 0$ in (10) is simply

$$K_0(z, w) = e^{\pi z \bar{w} - \frac{\pi}{2}(|z|^2 + |w|^2)},$$

which is the correlation kernel of the infinite Ginibre ensemble. Thus, the polyanalytic ensemble of the pure type associated with the first Landau level is, as mentioned in the introduction, the Ginibre ensemble.

Using the formula for the kernel in (10) and (5), we see that the *total correlation function of the polyanalytic ensemble of the pure type* is:

$$h_r(z) = -[L_r^0(\pi|z|^2)]^2 e^{-\pi|z|^2}, \quad z \in \mathbb{C}. \tag{11}$$

We note that h_r is a radial function and that $|h_r(z)| \leq C_\alpha e^{-\alpha\pi|z|^2}$, for every $\alpha \in (0, 1)$ —where C_α is a constant that depends on α .

4.2. Polyanalytic Fock spaces

A function $F(z, \bar{z})$, defined on a subset of \mathbb{C} , and satisfying the generalized Cauchy–Riemann equations

$$(\partial_{\bar{z}})^q F(z, \bar{z}) = \frac{1}{2^q} (\partial_x + i\partial_\xi)^q F(x + i\xi, x - i\xi) = 0, \tag{12}$$

is said to be *polyanalytic of order $q - 1$* [25]. It is clear from (12) that the following polynomial of order $q - 1$ in \bar{z}

$$F(z, \bar{z}) = \sum_{k=0}^{q-1} \bar{z}^k \varphi_k(z), \tag{13}$$

where the coefficients $\{\varphi_k(z)\}_{k=0}^{q-1}$ are analytic functions, is a polyanalytic function of order $q - 1$. By solving $\partial_{\bar{z}} F(z, \bar{z}) = 0$, an iteration argument shows that every $F(z, \bar{z})$ satisfying (12) is indeed of the form (13).

The polyanalytic Fock space $\mathbf{F}^q(\mathbb{C})$ consists of all the functions of the form $e^{-\frac{\pi}{2}|z|^2} F(z, \bar{z})$, with $F(z, \bar{z})$ polyanalytic functions of order $q - 1$, supplied with the Hilbert space structure of $L^2(\mathbb{C})$. The (infinite-dimensional) kernel of the polyanalytic Fock space $\mathbf{F}^q(\mathbb{C})$ is

$$\mathbf{K}^q(z, w) = L_q^1(\pi|z - w|^2) e^{\pi z \bar{w} - \frac{\pi}{2}(|z|^2 + |w|^2)}. \tag{14}$$

The connection to the Landau levels—see [14] for applications of this connection to signal analysis and [20, 26] to physics—follows from the following orthogonal decomposition, first observed by Vasilevski [27]:

$$\mathbf{F}^q(\mathbb{C}) = \mathcal{F}^0(\mathbb{C}) \oplus \dots \oplus \mathcal{F}^{q-1}(\mathbb{C}), \tag{15}$$

where $\mathcal{F}^r(\mathbb{C})$ is the pure Landau level eigenspace of order r , whence the terminology *pure* used in [20]. *Pure poly-Fock spaces* provide a full orthogonal decomposition of the whole $L^2(\mathbb{C})$:

$$L^2(\mathbb{C}) = \bigoplus_{r=1}^{\infty} \mathcal{F}^r(\mathbb{C}).$$

The formula for Laguerre polynomials $\sum_{r=0}^{q-1} L_r^\alpha = L_{q-1}^{\alpha+1}$ and (15) show that

$$\mathbf{K}^q(z, w) = \sum_{r=0}^{q-1} \mathcal{K}^r(z, w),$$

where $\mathcal{K}^r(z, w)$ is the reproducing kernel (10) of the pure Landau level eigenspace of order r .

5. The Weyl–Heisenberg ensembles

In this section we work with functions of several real variables, but keep a multi-index notation similar to the univariate case. As before, we let $d = 2m$ be an even positive integer.

5.1. The Schrödinger representation of the Heisenberg group

The infinite Weyl–Heisenberg ensembles are DPP’s associated with the representation of the Heisenberg group (in [13], finite-dimensional versions are investigated). Given a *window function* $g \in L^2(\mathbb{R}^m)$, the Schrödinger representation of the Heisenberg group \mathbb{H} acts on $L^2(\mathbb{R}^m)$ by means of the unitary operators

$$T(x, \xi, \tau)g(t) = e^{2\pi i\tau} e^{-\pi i x \xi} e^{2\pi i \xi t} g(t - x), \quad (x, \xi) \in \mathbb{R}^d, \tau \in \mathbb{R}.$$

The corresponding representation coefficients are

$$\langle f, T(x, \xi, \tau)g \rangle = e^{-2\pi i\tau} e^{\pi i x \xi} \langle f, e^{2\pi i \xi \cdot} g(\cdot - x) \rangle.$$

5.2. Time-frequency analysis

The short-time Fourier transform $V_g f(x, \xi)$ can be defined in terms of the above representation coefficients by eliminating the variable τ as follows:

$$V_g f(x, \xi) = e^{2\pi i\tau} e^{-\pi i x \xi} \langle f, T(x, \xi, \tau)g \rangle = \langle f, e^{2\pi i \xi \cdot} g(\cdot - x) \rangle.$$

We introduce convenient notation where we identify a pair $(x, \xi) \in \mathbb{R}^d$ with the complex vector $z = x + i\xi \in \mathbb{C}^m$. The *time-frequency shifts* of a function $g : \mathbb{R}^m \rightarrow \mathbb{C}$ are defined as follows:

$$\pi(z)g(t) := e^{2\pi i \xi t} g(t - x), \quad z = (x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m, \quad t \in \mathbb{R}^m.$$

With this notation, given a window function $g \in L^2(\mathbb{R}^m)$, the *short-time Fourier transform* of a function $f \in L^2(\mathbb{R}^m)$ with respect to g is

$$V_g f(z) := \langle f, \pi(z)g \rangle, \quad z \in \mathbb{R}^{2m}.$$

The subspace of $L^2(\mathbb{R}^{2m})$ which is the image of $L^2(\mathbb{R}^m)$ under the short-time Fourier transform with the window g ,

$$\mathcal{V}_g = \{V_g f : f \in L^2(\mathbb{R}^m)\} \subset L^2(\mathbb{R}^{2m}),$$

is a Hilbert space with reproducing kernel given by

$$K_g(z, w) = \langle \pi(w)g, \pi(z)g \rangle_{L^2(\mathbb{R}^m)}. \tag{16}$$

With the notation $z = (x, \xi), w = (x', \xi')$, the kernel can be written explicitly as

$$K_g(z, w) = \int_{\mathbb{R}^m} \overline{g(t-x)}g(t-x')e^{2\pi i t(\xi'-\xi)} dt. \tag{17}$$

We can now introduce the WH ensembles.

Definition 5.1. Let $g \in L^2(\mathbb{R}^m)$ be of norm 1 and such that

$$|V_g g(z)| \leq C(1+|z|)^{-s} < +\infty, \tag{18}$$

for some $s > 2m + 1$ and $C > 0$. The infinite Weyl–Heisenberg ensemble associated with the function $g \in L^2(\mathbb{R}^m)$ is the determinantal point process with correlation kernel

$$K_g(z, w) = \langle \pi(w)g, \pi(z)g \rangle_{L^2(\mathbb{R}^m)}.$$

Remark 5.2. The condition in (18) amounts to decay of g in both the space and frequency variables, and is satisfied by any Schwartz-class function.

Remark 5.3. The WH ensemble associated with a window g is well-defined due to the Macchi–Soshnikov theorem [28, 29]. Indeed, the kernel K_g represents a projection operator and we only need to verify that it is locally trace-class. Given a compact domain D , the operator $T_{g,D}$ represented by the localized kernel $K_{g,D}$ is known as a Gabor–Toeplitz operator. It is well-known that $T_{g,D}$ is trace-class and that $\text{trace}(T_{g,D}) = |D|$; see for example [15, 30].

Remark 5.4. For general windows g , the resulting WH ensemble is statistically anisotropic in the sense that the point-statistics may depend on the vector displacements between the points.

Figure 1 shows realizations of Weyl–Heisenberg ensembles corresponding to two different windows: the Gaussian and the Hermite function of order 7. As explained in section 5.4, these correspond to different Landau levels.

While the correlation kernel of a WH ensemble is not translationally invariant, a simple calculation shows that the corresponding point process is. In addition, using the explicit formula for the kernel in (17) and (5), we see that *the total correlation function of an infinite WH ensemble* is:

$$h_g(z) = - \left| \int_{\mathbb{R}^m} g(t-x)\overline{g(t)}e^{2\pi i t\xi} dt \right|^2 = -|V_g g(z)|^2, \quad z = (x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^d. \tag{19}$$

As we show in section 5.4, for concrete choices of the underlying window function g , it is possible to get explicit expressions for the corresponding function h_g . Moreover, in many cases, these are radial functions. We next show that, in that case, we can describe the asymptotics of the structure factor near the origin.

Lemma 5.5. *Assume that the correlation function h_g of a WH ensemble is radial. Then the corresponding structure factor satisfies:*

$$|S_g(\mathbf{k})| \asymp |\mathbf{k}|^2, \quad \text{as } \mathbf{k} \rightarrow 0.$$

More precisely, there exist two constants $c, C > 0$ such that $c|\mathbf{k}|^2 \leq |S_g(\mathbf{k})| \leq C|\mathbf{k}|^2$, for \mathbf{k} near 0.

Proof. Using (18) and (19), we see that $|h_g(z)| \lesssim (1 + |z|)^{-(2d+2)}$, and consequently

$$(1 + |z|^2) |h_g(z)| \in L^1(\mathbb{R}^d, dz).$$

Since $S_g(\mathbf{k}) = 1 + \hat{h}_g(\mathbf{k})$, it follows that the structure factor S_g is a C^2 function, and we can Taylor expand it as

$$S_g(\mathbf{k}) = S_g(0) + \sum_{j=1}^d \partial_{k_j} S_g(0) k_j + \sum_{|\alpha|=2} \partial_{\mathbf{k}}^\alpha S_g(0) k^\alpha + o(|\mathbf{k}|^2). \quad (20)$$

First, note that $S_g(0) = 1 + h_g(0) = 1 - \|g\|_2^2 = 0$. Second, since h_g is radial,

$$\partial_{k_j} S_g(0) = -i \int_{\mathbb{R}^d} z_j h_g(z) dz = 0,$$

and, therefore, the linear terms in (20) vanish. Similarly, the cross second derivatives in (20)— $\partial_{k_j} \partial_{k_{j'}} S_g(0)$, with $j \neq j'$ —vanish, leading to

$$S_g(\mathbf{k}) = \Delta_{\mathbf{k}} S_g(0) |\mathbf{k}|^2 + o(|\mathbf{k}|^2).$$

Hence, it suffices to show that $\Delta_{\mathbf{k}} S_g(0) \neq 0$. This is the case because

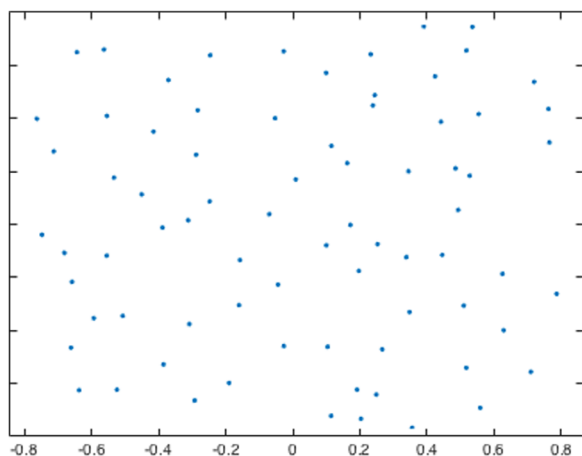
$$\Delta_{\mathbf{k}} S_g(0) = - \int_{\mathbb{R}^d} |z|^2 h_g(z) dz,$$

and $h_g \leq 0$, while $h_g \not\equiv 0$. □

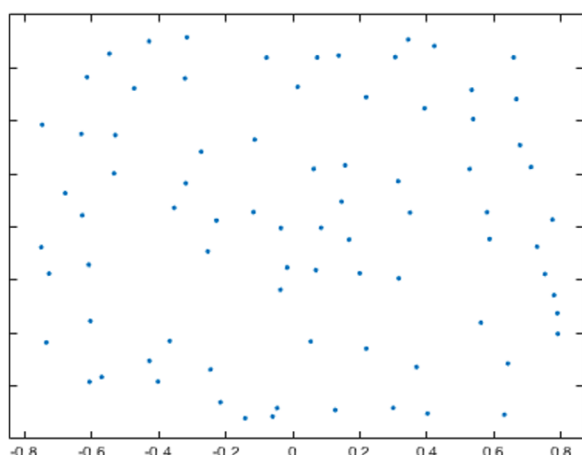
5.3. Hyperuniformity of infinite Weyl–Heisenberg ensembles

Now we will present our study of the variance of Weyl–Heisenberg ensembles, relying on spectral methods originating from time-frequency analysis. Given a set $\Omega \subseteq \mathbb{R}^d$, $\partial\Omega$ denotes its frontier, 1_Ω its characteristic function, $|\Omega|$ its measure and $|\partial\Omega|$ its perimeter (defined as the $d - 1$ dimensional measure of the boundary). The following is our main result.

The Weyl–Heisenberg ensemble: hyperuniformity and higher landau levels



(a)



(b)

Figure 1. WH ensembles corresponding to different Hermite windows. (a) Landau level #1. (b) Landau level #7.

Theorem 5.6. *Weyl–Heisenberg ensembles are hyperuniform. More precisely, if \mathcal{X} is a WH-ensemble and $D_R \subset \mathbb{R}^d$ is a d -dimensional ball of radius R , then, as $R \rightarrow \infty$,*

$$\sigma^2(R) = \mathbb{V}[\mathcal{X}(D_R)] \lesssim R^{d-1}.$$

Proof. We want to show that the number variance $\sigma^2(R) = \mathbb{E}[\mathcal{X}(D_R)^2] - \mathbb{E}[\mathcal{X}(D_R)]^2$ —where D_R is a $2m$ -dimensional ball of radius R —satisfies $\sigma^2(R) \lesssim R^{2m-1}$. We start defining the concentration operator

$$(T_{D_R}f)(z) = \int_{D_R} f(w)K_g(z, w)dw.$$

Using ([17, equation (1.2.4)]) one can write the number variance of $\mathcal{X}(D_R)$ as

$$\begin{aligned} \sigma^2(R) &= \mathbb{E} [\mathcal{X}(D_R)^2] - \mathbb{E} [\mathcal{X}(D_R)]^2 \\ &= \int_{D_R} K_g(z, z) dz - \int_{D_R \times D_R} |K_g(z, w)|^2 dzdw \\ &= \text{trace}(T_{D_R}) - \text{trace}(T_{D_R}^2) \\ &= |D_R| - \int_{D_R \times D_R} |K_g(z, w)|^2 dzdw. \end{aligned}$$

Now, one can use [15, proposition 3.4] to obtain the upper inequality for the number variance:

$$\sigma^2(R) \lesssim |\partial D_R| \int_{\mathbb{R}^{2m}} |z| |V_g g(z)|^2 dz \lesssim R^{2m-1}. \tag{21}$$

(See [31] for applications of this kind of inequalities to sampling theory.) □

Remark 5.7. The proof of theorem 5.6 extends to more general observation windows. In this case, the number variance is dominated by the perimeter of the observation window.

The next result shows that R^{d-1} is actually the precise rate of convergence.

Theorem 5.8. *The variance of a Weyl–Heisenberg ensemble satisfies, as $R \rightarrow \infty$,*

$$\sigma^2(R) = \mathbb{V}[\mathcal{X}(D_R)] \sim R^{d-1},$$

where D_R is a d -dimensional ball of radius R .

Proof. As we have seen in the proof of theorem 5.6,

$$\sigma^2(R) = \text{trace}(T_{D_R}) - \text{trace}(T_{D_R}^2).$$

Arguing as in the proof of lemma 3.3 in [15], we obtain the following formula, where the variance is bounded in terms of the counting function of the eigenvalues of $T_{D_R} - \{\lambda_k(R) : k \geq 1\}$ —that are above a certain threshold. More precisely, for $\delta \in (0, 1)$:

$$\sigma^2(R) = \text{trace}(T_{D_R}) - \text{trace}(T_{D_R}^2) \geq \frac{|\#\{k \geq 1 : \lambda_k(R) > 1 - \delta\} - |D_R||}{\max\{\frac{1}{\delta}, \frac{1}{1-\delta}\}}. \tag{22}$$

Now, by [32, theorem 4.1], there exists δ independent of R such that

$$|\#\{k \geq 1 : \lambda_k(R) > 1 - \delta\} - |D_R|| \gtrsim R^{2m-1}.$$

Combining this with (22) leads to the lower inequality

$$\sigma^2(R) \gtrsim R^{2m-1},$$

which, together with the upper inequality (21), yields the result. □

Remark 5.9. Theorem 5.8 implies that the central limit theorem of Costin and Lebowitz [33] (in the general formulation of Soshnikov [34]) is applicable and, therefore, the random variables $\mathcal{X}(D_R)$ —when properly rescaled—are asymptotically normal as $R \rightarrow \infty$.

Remark 5.10. Theorem 5.8 extends a result of Shirai [23], that concerns DPP’s that are translationally and rotationally invariant (with a suitably decaying correlation kernel). In this case, asymptotic formulas for the implied constants are also available. For general windows g , WH ensembles do not need to be rotationally invariant, see remark 5.4. It is noteworthy that the hyperuniformity concept has recently been generalized to incorporate anisotropic features [35] and thus the WH ensembles provide a rigorous testbed to study directional hyperuniformity.

5.4. Weyl–Heisenberg ensembles for higher Landau levels: polyanalytic Ginibre-type ensembles

For $m = 1$, using the notation $z = x + i\xi$ and $w = u + i\eta$, a calculation (see [36]) shows that the reproducing kernel of \mathcal{V}_{h_r} is related as follows to the reproducing kernel of the pure Fock space of polyanalytic functions:

$$K_{h_r}(\bar{z}, \bar{w}) = e^{-i\pi(u\eta - x\xi) - \pi \frac{|z|^2 + |w|^2}{2}} L_r^0(\pi |z - w|^2) e^{\pi z \bar{w}}. \tag{23}$$

Thus, the operator E which maps F to

$$e^{-i\pi x \xi} F(\bar{z})$$

is an isometric isomorphism

$$E : \mathcal{V}_{h_r} \rightarrow \mathcal{F}^r(\mathbb{C}).$$

Thus, all properties of Weyl–Heisenberg ensembles are automatically translated to the polyanalytic ensembles, in particular the hyperuniformity property. In addition, polyanalytic ensembles, as presented in section 4, extend verbatim to $\mathbb{C}^m = \mathbb{R}^d$, provided that the formulae are interpreted in a vectorial sense. With this understanding, we obtain from theorem 5.6 the following corollary.

Corollary 5.11. *The pure polyanalytic ensembles are hyperuniform, and, as $R \rightarrow \infty$,*

$$\sigma^2(R) = \mathbb{V}[\mathcal{X}(D_R)] \sim R^{d-1},$$

where D_R is a d -dimensional ball of radius R .

In the case $d = 2$, this has been proved in [23, theorem 1] using explicit computations which also provide the value of the asymptotic constant for the pure polyanalytic Ginibre ensemble of order r , $C_r = \frac{8}{\pi^2} r^{1/2}$. As noted in remark 5.7, the variance bounds in theorem 5.6 also apply to more general observation windows, and these conclusions therefore extend to the polyanalytic ensembles.

6. Conclusions

We introduced the infinite Weyl–Heisenberg ensembles in \mathbb{R}^d and showed that they are hyperuniform. This provides another class of examples of d -dimensional determinantal

point processes that are hyperuniform beyond the so-called Fermi-type varieties [4]. We also proved that the number variance associated with spherical observation windows of radius R grows like the surface area of the window, R^{d-1} . Due to the Costin–Lebowitz central limit theorem, this implies that the number of particles of a WH ensemble within a growing observation window are asymptotically normal random variables.

We gave explicit formulas for the total correlation functions of WH ensembles. In the radial case, we also derived asymptotics near the origin for the structure factor, and showed that $S(k) \asymp k^2$ in the limit $k \rightarrow 0$ in all space dimensions. The two-dimensional point process associated with the Ginibre ensemble has similar quadratic in k structure-factor asymptotics.

Special choices of the waveform g in the definition of a WH ensemble lead to important point processes. Specifically, we showed that when g is chosen as a Hermite function, the corresponding point process coincides with the so-called polyanalytic Ginibre ensemble of the pure type, which models the distribution of electrons in higher Landau levels. The corresponding total correlation functions resemble the ones of the Ginibre ensemble, this time with a Laguerre polynomial as an additional multiplicative factor. In particular, they decay, for large distances, faster than exponential; specifically, like a Gaussian.

The family of Weyl–Heisenberg ensembles also includes processes that are *structurally anisotropic* in the sense that the point-statistics depend on the different spatial directions. Thus, our work provides the first rigorous means to study *directional hyperuniformity* of point processes. In such instances, it is relevant to consider a more general notion of hyperuniformity that accounts for the dependence of the structure factor on the direction in which the origin in Fourier space is approached [35].

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